# MATH 103B – Discussion Worksheet 2
**April 13, 2023**

**Topics**: Rings and ideals (Judson 16.1-16.3)

## Supplement to Lectures

Let $R$ be a ring. For any nonnegative integer $n$ and any element $r \in R$, we write $r + \cdots + r(n$ times) as $nr$.

**Definition 0.1.** The **characteristic** of a ring $R$ is the least positive integer $n$ such that $nr = 0$ for all $r \in R$.

**Theorem 0.2** (Judson Theorem 16.19)**.** *The characteristic of any integral domain is either prime or zero.*

Recall the definition of a unit in a ring $R$. Let $R$ be a ring with unity (i.e. a ring with a multiplicative identity 1), let $R^* = \{a \in R | a \text{ is a unit}\}$. Then $R^*$ is a group (under multiplication).

**Example 0.1.** $\mathbb{Z}^* = \{\pm 1\}$.

**Example 0.2** (Judson Example 16.12)**.** $\mathbb{Z}[i]^* = \{\pm 1, \pm i\}$.

## Discussion Problems

Recall the definition of ideals in a ring.

**Problem 1.** Let $R$ be a commutative ring with unity. Consider the ideal $I = (x, y)$ in $R[x, y]$. List three elements in $R[x, y]$ that are in $I$ and three elements that are not in $I$. Then prove that $I$ is indeed an ideal, and that $I$ is not principal.

**Problem 2.** Let $R$ be a ring with unity and $I$ an ideal in $R$. Suppose $1 \in I$. Prove $I = R$.

**Problem 3.** Prove $(0)$ is always an ideal in any ring $R$.

**Problem 4.** Let $R$ be a ring. Show that $\varphi : R[x] \to R$ defined by $\varphi(f) = f(0)$ is a *ring* homomorphism. Compute $\ker \varphi$. Suppose $\varphi$ is instead defined by $\varphi(f) = f(1)$. Is $\varphi$ a ring homomorphism.

**Problem 5.** Let $R = \mathbb{Z}[\sqrt{-5}]$. Consider $I = (2, 1 + \sqrt{-5}) \subseteq R$. Determine whether the following elements in $R$ are in $I$: $3, 6, 1, \sqrt{-5}$.
(Bonus) Prove or disprove: $I$ is a principal ideal in $R$.